

A new treatment for some periodic Schrödinger operators II: the wave function

Wei He*

*School of Electronic Engineering,
Chengdu Technological University, Chengdu 611730, China*

Abstract

Following the approach of our previous paper we continue to study the asymptotic solution of periodic Schrödinger operators. Using the eigenvalues obtained earlier the corresponding asymptotic wave functions are derived. This gives further evidence in favor of the monodromy relations for the Floquet exponent proposed in the previous paper. In particular, the large energy asymptotic wave functions are related to the instanton partition function of $N=2$ supersymmetric gauge theory with surface operator. A relevant number theoretic dessert is appended.

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*weihephys@foxmail.com

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1 Introduction

The study of the Schrödinger operator with elliptic potentials, which appear in mechanics, spectral theory, integrable system and some quantum field theory models, has been persistent since 19th century. Nevertheless, we still lack a thorough understanding of its solution. In a previous paper [1] we introduced a new method to obtain the asymptotic eigenvalue of some periodic Schrödinger operators. In particular, we revised the Floquet theory for the elliptic potentials, proposed new monodromy relations in order to obtain eigenvalues consistent with known results in the literature. Our approach is inspired by the ideas and methods from supersymmetric quantum gauge theory, it has not yet achieved a satisfying mathematical foundation. However, before the achievement of a mathematical rigorous understanding, it should not bother us to extract more results.

Following this method, it is very convenient to derive the corresponding asymptotic wave functions, we present the results in this paper. We focus on our canonical examples: the Mathieu equation and the Lamé equation. The relevant literature are extensive, some wave functions has been studied before, we briefly comment the old materials where earlier treatments can be found. The eigenfunction (46) has not been studied before. We need to use the asymptotic eigenvalues obtained in [1] to derive the corresponding wave function, therefore this provides another consistency evidence for the method we use. The treatment

is systematic, it could be applied to other similar periodic potentials whose eigenvalues are analyzed in [1].

In this paper we focus on the formal asymptotic solutions, we are not concerned much about some special properties that are required in quantum mechanics, because these periodic spectral problem also appear in broader circumstances. The interpretation of the corresponding solution depends on the context where the problem arises. Indeed, these periodic potentials arises in a relation between quantum integrable models and supersymmetric gauge theory [10] where the methods and interpretations significantly deviate the familiar routine. Motivated by this connection we have studied how the gauge theory could help to derive the eigenvalues [11, 1], here we show that the eigenfunctions are also related to gauge theory.

2 Large energy wave function

2.1 The general method

For a high energy particle, any smooth time independent potential of finite strength could be treated as small perturbation. The problem we treat in the nonrelativistic regime is to solve the asymptotic spectral problem for the following 1-dimensional stationary Schrödinger equation with periodic potential,

$$(\partial_x^2 - u(x))\psi = \lambda\psi, \quad u(x) = u(x + T). \quad (1)$$

The potential might be multi-components, each with a coupling constant g . The strength of the potential can be described by a characteristic strength κ which is the dominant one among all g or certain average of all g , see examples in [1]. When the eigenvalue λ is very large, $\lambda \gg \kappa$, an asymptotic solution exists. The large energy solution discussed in this section is also the weak coupling perturbative solution.

Recall how we derive the large energy dispersion relation [1]. We write the wave function in the form

$$\psi(x) = \exp\left(\int^x v(y)dy\right), \quad (2)$$

then $v(x)$ satisfies the relation

$$v_x + v^2 = u + \lambda. \quad (3)$$

For large λ we can expand $v(x)$ in the following form

$$v(x) = \sqrt{\lambda} + \sum_{\ell=1}^{\infty} \frac{v_{\ell}(x)}{(\sqrt{\lambda})^{\ell}}. \quad (4)$$

Then $v_\ell(x)$ can be solved recursively, they are given by the KdV Hamiltonian densities,

$$\begin{aligned} v_1 &= \frac{1}{2}u, & v_2 &= -\frac{1}{4}u_x, \\ v_3 &= -\frac{1}{8}(u^2 - u_{xx}), & v_4 &= \frac{1}{16}(2u^2 - u_{xx})_x, \quad \text{etc.} \end{aligned} \quad (5)$$

We use the notation $u_x = \partial_x u$, $u_{xx} = \partial_x^2 u$, etc.

In fact there is another solution for the relation (3), it is expanded in the same form of (4) with $\sqrt{\lambda}$ changed to $-\sqrt{\lambda}$. Therefore we get the unnormalized asymptotic expansion of the corresponding wave functions,

$$\psi_\pm(x) = \exp\left(\pm \lambda^{\frac{1}{2}}x \pm \frac{1}{2\lambda^{\frac{1}{2}}} \int^x u(y)dy - \frac{1}{4\lambda}u(x) \mp \frac{1}{8\lambda^{\frac{3}{2}}}\left[\int^x u^2(y)dy - u_x\right] + \dots\right). \quad (6)$$

The dispersion relation $\lambda(\nu)$, where ν is the Floquet exponent, is obtained by the Floquet property of the wave function,

$$\exp\left(\int_x^{x+T} v(y)dy\right) = \exp(\pm i\nu T). \quad (7)$$

The relations (6) and (7) give a complete perturbative solution for large energy.

2.2 Mathieu equation

The potential for the Mathieu equation is

$$u(x) = 2h \cos 2x, \quad (8)$$

from (6) we get the large energy asymptotic wave functions

$$\psi_\pm(x) = \exp\left(\pm \lambda^{\frac{1}{2}}x \pm \frac{h \sin 2x}{2\lambda^{\frac{1}{2}}} - \frac{h \cos 2x}{2\lambda} \mp \frac{8h \sin 2x + 4h^2x + h^2 \sin 4x}{16\lambda^{\frac{3}{2}}} + \dots\right). \quad (9)$$

It is clear that with λ as the expansion parameter, the coefficients of $\lambda^{-\frac{\ell}{2}}$ are not necessary periodic functions. However, the wave functions must satisfy the Floquet property, this is made clear by the following parameter change using the eigenvalue expansion.

The large energy dispersion relation is a classic result, see for example [8]. Formula (7) provides another convenient way to compute [1], we have

$$\lambda = -\nu^2 - \frac{h^2}{2\nu^2} - \frac{h^2}{2\nu^4} - \frac{16h^2 + 5h^4}{32\nu^6} + \dots. \quad (10)$$

It has a solution

$$\lambda^{\frac{1}{2}} = i(\nu + \frac{h^2}{4\nu^3} + \frac{h^2}{4\nu^5} + \frac{16h^2 + 3h^4}{64\nu^7} + \dots). \quad (11)$$

Another solution $-\lambda^{\frac{1}{2}}$ does not lead to new wave function. Substitute $\lambda^{\frac{1}{2}}$ into (9) we get the wave functions in the form

$$\psi_{\pm}(x) = \exp \left(\pm i\nu x \mp \frac{ih \sin 2x}{2\nu} + \frac{h \cos 2x}{2\nu^2} \mp \frac{i(8h \sin 2x + h^2 \sin 4x)}{16\nu^3} + \dots \right). \quad (12)$$

Now the coefficients of ν^{-l} , with $l \geq 1$, are periodic functions, the wave functions take the form $\psi_{\pm}(x) = e^{\pm i\nu x} \phi(\pm x)$ with $\phi(x)$ a periodic function.

Another bases of the asymptotic wave functions, commonly used in many literatures [2, 3, 4, 5, 6, 7], are $ce_m(x)$ and $se_m(x)$. Up to a constant, their relation to $\psi_{\pm}(x)$ is

$$\psi_{\pm}(x)|_{\nu=m} \sim ce_m(x) \pm ise_m(x). \quad (13)$$

2.3 Lamé equation

The potential for the Lamé equation is an elliptic function, as we have explained in [1], for the large energy perturbation we should use the Weierstrass form to compute. In this paper we use the potential

$$u(x) = \alpha \tilde{\wp}(x; 2\omega_1, 2\omega_2), \quad (14)$$

which is define by a shifted elliptic function $\tilde{\wp}(x; 2\omega_1, 2\omega_2) = \wp(x; 2\omega_1, 2\omega_2) + \zeta_1$ with ζ_1 a constant given by the Weierstrass zeta function $\zeta_1 = \frac{\zeta(\omega_1)}{\omega_1}$. The coupling constant α is often represented as $n(n-1)$, the nome of the elliptic function is $q = \exp(2\pi i \frac{\omega_2}{\omega_1})$. The use of shifted potential is more convenient for us to see the relation to N=2 supersymmetric gauge theory in the Section 4, it does not change essential aspects of the spectral solution. There are three periods $2\omega_1, 2\omega_2$ and $2\omega_3 = 2\omega_1 + 2\omega_2$ play a role in the asymptotic spectral solution. The large energy asymptotic solution is related to the period $2\omega_1$, the corresponding wave functions are

$$\psi_{\pm}(x) = \exp \left(\pm \lambda^{\frac{1}{2}} x \mp \frac{\alpha \tilde{\zeta}(x)}{2\lambda^{\frac{1}{2}}} - \frac{\alpha \tilde{\wp}(x)}{4\lambda} \mp \frac{2\alpha(\alpha-6)\tilde{\wp}_x - 24\alpha^2\zeta_1\tilde{\zeta}(x) + \alpha^2(g_2 - 12\zeta_1^2)x}{96\lambda^{\frac{3}{2}}} + \dots \right), \quad (15)$$

where g_2, g_3 are the modular invariants of $\wp(x)$. We use another shifted function defined by $\tilde{\zeta}(x) = \zeta(x) - \zeta_1 x$, it is $2\omega_1$ -periodic and satisfies the relation $\partial_x \tilde{\zeta}(x) = -\tilde{\wp}(x)$. Again, some coefficients of $\lambda^{-\frac{n}{2}}$ are not periodic functions.

The corresponding dispersion relation was derived by E. Langmann [9], expressed as a q -series, the same expression also appears in the context of its relation to gauge theory [10] as we show in [11]. Another way to derive the dispersion relation is to use the formula (7), then we get an expression involving quasi-modular functions [12, 1],

$$\lambda = -\nu^2 + \frac{\alpha^2(12\zeta_1^2 - g_2)}{48\nu^2} + \frac{\alpha^3(20\zeta_1^3 - g_2\zeta_1 - g_3) - \alpha^2(2g_2\zeta_1 - 3g_3)}{80\nu^4} + \dots \quad (16)$$

It has a solution

$$\lambda^{\frac{1}{2}} = i \left(\nu - \frac{\alpha^2(12\zeta_1^2 - g_2)}{96\nu^3} - \frac{\alpha^3(20\zeta_1^3 - g_2\zeta_1 - g_3) - \alpha^2(2g_2\zeta_1 - 3g_3)}{160\nu^5} + \dots \right). \quad (17)$$

Substitute $\lambda^{\frac{1}{2}}$ into (15) we get the wave functions in the Floquet form,

$$\psi_{\pm}(x) = \exp \left(\pm i\nu x \pm \frac{i\alpha\tilde{\zeta}(x)}{2\nu} + \frac{\alpha\tilde{\wp}(x)}{4\nu^2} \pm \frac{i[12\alpha^2\zeta_1\tilde{\zeta}(x) - \alpha(\alpha - 6)\tilde{\wp}_x]}{48\nu^3} + \dots \right). \quad (18)$$

The wave functions also satisfy the property $\psi_{\pm}(-x) = \psi_{\mp}(x)$.

There is a comment about the polynomials of elliptic functions that appear in (15) and (18). Recall that any elliptic function can be expressed as a linear combination of zeta functions and their derivatives. In fact, the Hamiltonian densities $v_{\ell}(x)$ for the elliptic potential have no pole of order one at $x = 0$, therefore, they are linear combinations of $\partial_x^k \tilde{\wp}(x)$ with $k \geq 0$. Then the integrated Hamiltonians appearing in (15) are linear combinations of x and $\partial_x^k \tilde{\zeta}(x)$ with $k \geq 0$. In the wave function (18) the phase $e^{\pm i\nu x}$ contains the linear term of x , the coefficients of ν^{-l} are linear combinations of $\partial_x^k \tilde{\zeta}(x)$ with $k \geq 0$, probably include a x -independent constant term. The constant terms can be absorbed into the normalization constant, then the expressions are linear polynomials of $\partial_x^k \tilde{\zeta}(x)$ with $k \geq 0$. Or equivalently, because $\tilde{\zeta}(x) = \partial_x \ln \vartheta_1(\frac{\pi x}{2\omega_1}, q)$, they are linear polynomials of $\partial_x^k \ln \vartheta_1(\frac{\pi x}{2\omega_1}, q)$ with $k \geq 1$. This point is important when we connect the wave function to the instanton partition function in the Section 4, especially for higher order terms which we do not explicitly give in (18).

When take the limit $q \rightarrow 0, \alpha \rightarrow \infty$ with $\alpha q^{\frac{1}{2}} \rightarrow -\frac{h}{4}$ fixed, all the results obtained here reduce to the case of the Mathieu equation. Some details in the limit process need further explanation. We first examine how the elliptic potential is reduced to the trigonometric potential. Instead of taking the limit for $\tilde{\wp}(x)$, we have to shift the argument and take the limit for $\tilde{\wp}(x + \omega_2)$. From the leading order expansion of $\lim_{q \rightarrow 0} \tilde{\wp}(x + \omega_2)$ given below in (21), the resultant potential takes the form

$$u(x) = 2h \left(\frac{\pi}{2\omega_1} \right)^2 \cos \frac{\pi x}{\omega_1}, \quad (19)$$

with period $2\omega_1$, its eigenvalue denoted by $\tilde{\lambda}$ can be obtained from the limit of (16). If we use the rescaled coordinate $\chi = \frac{\pi x}{2\omega_1}$ and eigenvalue $\lambda = \left(\frac{2\omega_1}{\pi} \right)^2 \tilde{\lambda}$, the limit of the Lamé equation could be written in the standard form of the Mathieu equation,

$$(\partial_{\chi}^2 - 2h \cos 2\chi)\psi = \lambda\psi. \quad (20)$$

The corresponding large energy asymptotic eigenvalue λ and wave functions $\psi_{\pm}(\chi, \nu)$ take the same functional form as the eigenvalue (10) and the wave functions (12), but with the coordinate variable substituted by χ .

Let us inspect more carefully the limit for eigenvalue and wave functions of the Lamé equation. For the wave functions (18), we shift the argument by $x \rightarrow x + \omega_2$, then take the limit $q \rightarrow 0$ (i.e. with ω_1 fixed, $\omega_2 \rightarrow i\infty$) for $\psi_{\pm}(x + \omega_2)$. The following expansions are needed,

$$\lim_{q \rightarrow 0} \tilde{\wp}(x + \omega_2) = \left(\frac{\pi}{2\omega_1}\right)^2 \left(-8q^{\frac{1}{2}} \cos 2\chi - 16q \cos 4\chi - 8q^{\frac{3}{2}}(\cos 2\chi + 3 \cos 6\chi) + \mathcal{O}(q^2) \right), \quad (21)$$

$$\lim_{q \rightarrow 0} \tilde{\zeta}(x + \omega_2) = \frac{\pi}{2\omega_1} \left(4q^{\frac{1}{2}} \sin 2\chi + 4q \sin 4\chi + 4q^{\frac{3}{2}}(\sin 2\chi + \sin 6\chi) + \mathcal{O}(q^2) \right). \quad (22)$$

The Lamé wave functions (18) indeed reduce to the Mathieu wave functions, iff we further substitute the Floquet exponent by $\tilde{\nu} = \frac{\pi}{2\omega_1}\nu$. The modification of the exponent can be understood as follows. In this subsection, the exponent ν conjugates to the variable χ of the π -periodic potential $\cos 2\chi$, it is different from the exponent that conjugates to the variable x of the $2\omega_1$ -periodic potential $\cos \frac{\pi x}{2\omega_1}$ which we should have denoted by another letter $\tilde{\nu}$. According to the Floquet theorem their wave functions produce phases under periodic shift by $\psi_{\pm}(\chi + \pi, \nu) = \exp(\pm i\nu\pi)\psi_{\pm}(\chi, \nu)$ and $\psi_{\pm}(x + 2\omega_1, \tilde{\nu}) = \exp(\pm i2\tilde{\nu}\omega_1)\psi_{\pm}(x, \tilde{\nu})$. But actually the potentials are the same, so the phases must be the same, which means the exponents are related by the relation $\nu\pi = 2\tilde{\nu}\omega_1$. It is easy to check that the limit for the eigenvalue (16) which is associated to the potential $\cos \frac{\pi x}{2\omega_1}$, hence with ν substituted by $\tilde{\nu}$, equals $(\frac{\pi}{2\omega_1})^2\lambda$ where λ is the Mathieu eigenvalue for the potential $\cos 2\chi$.

3 Small energy wave function

3.1 Location of small energy perturbation

Besides the large energy solution, there exists other solutions which are small energy excitations around local minima, i.e. the critical points of potential. We do not have a definite answer about if each local minimum gives an asymptotic solution, however, for some periodic potentials all known asymptotic solutions are located at a local minimum [1].

For example, the potential $u(x) = 2h \cos 2x$ has local minima at $x_* = 0$ and $x_* = \frac{\pi}{2}$ modulo periods. At the minima $u(x_*) = \pm 2h$, therefore the eigenvalues take the form

$$\lambda = \mp 2h + \delta, \quad (23)$$

where δ is the energy of small excitations. The small energy perturbation is also the strong coupling expansion for the potential, $h \gg 1$. In a similar way, the elliptic potential $u(x) = \alpha \tilde{\wp}(x; 2\omega_1, 2\omega_2)$ has local minima at $x_* = \omega_i$, at which the potential $u(x_*) = \alpha(e_i + \zeta_1)$, $i = 1, 2, 3$. The first minima at $x_* = \omega_1$ actually corresponds to the large energy excitations

already analyzed in the section 2.3, the leading order energy comes from the quasimomentum $\lambda \sim -\nu^2 + \dots$. The other two minima indeed give small energy perturbative solutions, nevertheless, we should use the Jacobian form of the Lamé equation to compute, as explained in [1].

3.2 Mathieu equation

The first small energy perturbation

Around the minimum $x_* = 0$, $\lambda = -2h + \delta$, the potential strength $h \gg 1$ is large compared to the energy δ , therefore the expansion parameter is $h^{\frac{1}{2}}$. The relation $v_x + v^2 = u + \lambda$ has an asymptotic solution in the form [1]

$$v(x) = \sum_{\ell=-1}^{\infty} \frac{v_{\ell}(x)}{(\sqrt{h})^{\ell}}, \quad (24)$$

with

$$\begin{aligned} v_{-1} &= 2i \sin x, \\ v_0 &= -\frac{1}{2} \cot x, \\ v_1 &= \frac{i}{16} \csc x (\cot^2 x + 2 \csc^2 x - 4\delta), \\ v_2 &= \frac{1}{32} \csc^2 x \cot x (\cot^2 x + 5 \csc^2 x - 4\delta), \quad \text{etc.} \end{aligned} \quad (25)$$

This leads to the following asymptotic wave functions,

$$\begin{aligned} \psi_{\pm}(x) &= \exp \left(\pm 2ih^{\frac{1}{2}} \cos x - \frac{1}{2} \ln \sin x \pm \frac{i}{2^5 h^{\frac{1}{2}}} [(8\delta - 1) \ln \tan \frac{x}{2} + 3 \csc x \cot x] \right. \\ &\quad \left. - \frac{1}{2^7 h} (\cot^4 x + 5 \csc^4 x - 8\delta \csc^2 x) + \dots \right). \end{aligned} \quad (26)$$

We can change the parameter δ to the Floquet exponent ν , by the following strong coupling expansion of the dispersion relation which is well known [8],

$$\lambda = -2h + 4\nu h^{\frac{1}{2}} - \frac{4\nu^2 - 1}{2^3} - \frac{4\nu^3 - 3\nu}{2^6 h^{\frac{1}{2}}} - \frac{80\nu^4 - 136\nu^2 + 9}{2^{12} h} + \dots \quad (27)$$

Then we get the wave functions in the form

$$\begin{aligned} \psi_{\pm}(x) &= (\sin \frac{x}{2})^{\pm i\nu - \frac{1}{2}} (\cos \frac{x}{2})^{\mp i\nu - \frac{1}{2}} \exp \left(\pm i2h^{\frac{1}{2}} \cos x + \frac{\csc^2 x}{2^5 h^{\frac{1}{2}}} [4\nu \pm i(3 - 4\nu^2) \cos x + 4\nu \cos 2x] \right. \\ &\quad \left. - \frac{\csc^4 x}{2^{10} h} [42 - 124\nu^2 \mp i(155\nu - 36\nu^3) \cos x + 4 \cos 2x \pm i(3\nu - 4\nu^3) \cos 3x \right. \\ &\quad \left. + 2(1 - 2\nu^2) \cos 4x] + \dots \right). \end{aligned} \quad (28)$$

Because the wave functions are unnormalized, the terms in the exponent might appear in slightly different form, nevertheless the differences are constants and can be absorbed into the normalization constant. This comment applies to all of the asymptotic wave functions in this paper. When the exponent ν takes real value the asymptotic wave functions have the property $\psi_{\pm}^*(x) = \psi_{\mp}(x)$. This solution seems often not recorded in the mathematical literature, however, it was analyzed in a paper by M. Stoner and J. Reeve [13]. The book [14] contains a discussion about this solution in the context of quantum mechanics.

The second small energy perturbation

Around the minimum $x_* = \frac{\pi}{2}$, $\lambda = 2h + \delta$, the potential strength $h^{\frac{1}{2}}$ again serves as the expansion parameter. The relation $v_x + v^2 = u + \lambda$ has an asymptotic solution in the form (24), with

$$\begin{aligned} v_{-1} &= 2 \cos x, \\ v_0 &= \frac{1}{2} \tan x, \\ v_1 &= -\frac{1}{16} \sec x (\tan^2 x + 2 \sec^2 x - 4\delta), \\ v_2 &= \frac{1}{32} \sec^2 x \tan x (\tan^2 x + 5 \sec^2 x - 4\delta), \quad \text{etc.} \end{aligned} \tag{29}$$

The corresponding wave functions are

$$\begin{aligned} \psi_{\pm}(x) &= \exp \left(\pm 2h^{\frac{1}{2}} \sin x - \frac{1}{2} \ln \cos x \pm \frac{1}{2^5 h^{\frac{1}{2}}} [(8\delta - 1) \ln \frac{\cos \frac{x}{2} + \sin \frac{x}{2}}{\cos \frac{x}{2} - \sin \frac{x}{2}} - 3 \sec x \tan x] \right. \\ &\quad \left. + \frac{1}{2^7 h} (\tan^4 x + 5 \sec^4 x - 8\delta \sec^2 x) + \dots \right). \end{aligned} \tag{30}$$

The dispersion relation at this local minimum [8]

$$\lambda = 2h - 4\nu h^{\frac{1}{2}} + \frac{4\nu^2 + 1}{2^3} + \frac{4\nu^3 + 3\nu}{2^6 h^{\frac{1}{2}}} + \frac{80\nu^4 + 136\nu^2 + 9}{2^{12} h} + \dots \tag{31}$$

allows us to change the parameter δ to the Floquet exponent ν . Then we get the corresponding asymptotic wave functions

$$\begin{aligned} \psi_{\pm}(x) &= (\sec x)^{\nu + \frac{1}{2}} \left[\frac{\cos(\frac{x}{2} + \frac{\pi}{4})}{\sin(\frac{x}{2} + \frac{\pi}{4})} \right]^{2\nu} \exp \left(\pm 2h^{\frac{1}{2}} \sin x + \frac{\sec^2 x}{2^5 h^{\frac{1}{2}}} [8\nu \mp (3 + 4\nu^2) \sin x] \right. \\ &\quad \left. + \frac{\sec^4 x}{2^{10} h} [39 + 112\nu^2 \mp (155\nu + 36\nu^3) \sin x \mp (3\nu + 4\nu^3) \sin 3x \right. \\ &\quad \left. - 8(1 + 2\nu^2) \cos 2x + \cos 4x] + \dots \right). \end{aligned} \tag{32}$$

The study of this solution dates back to the work of E. Ince and the the work of S. Goldstein in the 1920s. Some recent materials easier to access include the paper by R. Dingle and H. Müller [15], the book by F. Arscott [4], the book by H. Müller-Kirsten [14].

This asymptotic solution is related to the large h limit of the standard Mathieu functions by

$$(\psi_+ + \psi_-)|_{\nu=m+\frac{1}{2}} \sim ce_m(x), \quad (\psi_+ - \psi_-)|_{\nu=m+\frac{1}{2}} \sim se_{m+1}(x), \quad (33)$$

with m takes either even or odd integers [4, 7]. As we have $\psi_{\pm}(-x) = \psi_{\mp}(x)$, then $ce_m(x)$ is an even function and $se_m(x)$ is an odd function, as desired.

3.3 Lamé equation

Now we turn to the more interesting case of elliptic potential where the advantage of our method becomes more transparent. As we have shown in [11, 1], for small energy perturbative solution the Jacobian form of the elliptic function is more suitable, therefore we rewrite the potential as $u(z) = \alpha k^2 \text{sn}^2(z|k^2)$, and the Lamé equation is

$$(\partial_z^2 - \alpha k^2 \text{sn}^2 z)\psi = \Lambda \psi. \quad (34)$$

To transform the elliptic functions from $\tilde{\varphi}(x)$ to $\text{sn}^2 z$ we use relations of the coordinates, eigenvalues, the elliptic modulus and the nome,

$$x = \frac{z + iK'}{(e_1 - e_2)^{1/2}}, \quad \lambda = (e_1 - e_2)\Lambda - (e_2 + \zeta_1)\alpha, \quad k^2 = \frac{\vartheta_2^4(q)}{\vartheta_3^4(q)}. \quad (35)$$

We also will use μ to denote the Floquet exponent throughout of this subsection, the reason is explained in [11, 1].

The locations of the small energy perturbations are given by two solutions of the condition $\partial_z \text{sn}^2 z = 0$ at $z_* = 0$ and $z_* = K$ which correspond to $u(z_*) = 0$ and $u(z_*) = \alpha k^2$.

The first small energy expansion

Around $z_* = 0$ we have $\Lambda = 0 + \text{small correction}$, therefore in this case Λ itself is a small quantity compared with the potential strength αk^2 , and could serve as the expansion parameter. The relation $v_z + v^2 = u + \Lambda$ has an asymptotic solution that can be expanded as

$$v(z) = \sum_{\ell=-1}^{\infty} \frac{v_{\ell}(z)}{(\sqrt{\alpha})^{\ell}}, \quad (36)$$

with

$$\begin{aligned} v_{-1} &= k \text{sn} z, \\ v_0 &= -\frac{1}{2} \partial_z \ln \text{sn} z, \\ v_1 &= \frac{1}{8} k \text{sn} z + \frac{4\Lambda + 1 + k^2}{8k \text{sn} z} - \frac{3}{8k \text{sn}^3 z}, \\ v_2 &= -\frac{1}{16k^2} \partial_z \left(\frac{4\Lambda + 1 + k^2}{\text{sn}^2 z} - \frac{3}{\text{sn}^4 z} \right), \quad \text{etc.} \end{aligned} \quad (37)$$

By a straightforward integration we get the following asymptotic wave functions,

$$\begin{aligned} \psi_{\pm}(z) = \exp \Big(& \pm \alpha^{\frac{1}{2}} \ln(\text{dn}z - k\text{cn}z) - \frac{1}{2} \ln \text{sn}z \\ & \pm \frac{1}{2^4 \alpha^{\frac{1}{2}} k} \left[\frac{3\text{cn}z \text{dn}z}{\text{sn}^2 z} + 2k \ln(\text{dn}z - k\text{cn}z) - (8\Lambda - 1 - k^2) \ln \frac{\text{dn}z + \text{cn}z}{\text{sn}z} \right] \\ & + \frac{3 - (4\Lambda + 1 + k^2)\text{sn}^2 z}{2^4 \alpha k^2 \text{sn}^4 z} + \dots \Big). \end{aligned} \quad (38)$$

In order to change the parameter Λ to the Floquet exponent μ , we use the widely known strong coupling expansion of the dispersion relation [4, 8, 14]

$$\begin{aligned} \Lambda = & -i2\alpha^{\frac{1}{2}} k \mu - \frac{1}{2^3} (1 + k^2)(4\mu^2 - 1) \\ & - \frac{i}{2^5 \alpha^{\frac{1}{2}} k} [(1 + k^2)^2 (4\mu^3 - 3\mu) - 4k^2 (4\mu^3 - 5\mu)] \\ & + \frac{1}{2^{10} \alpha k^2} (1 + k^2)(1 - k^2)^2 (80\mu^4 - 136\mu^2 + 9) + \dots \end{aligned} \quad (39)$$

It leads to the wave functions in the form

$$\begin{aligned} \psi_{\pm}(z) = \exp \Big(& \pm \alpha^{\frac{1}{2}} \ln(\text{dn}z - k\text{cn}z) - \frac{1}{2} (\ln \text{sn}z \mp 2i\mu \ln \frac{\text{dn}z + \text{cn}z}{\text{sn}z}) \\ & \pm \frac{1}{2^4 \alpha^{\frac{1}{2}} k} \left[\frac{\pm 8i\mu + (3 - 4\mu^2)\text{cn}z \text{dn}z}{\text{sn}^2 z} + 2k \ln(\text{dn}z - k\text{cn}z) \right] \\ & + \frac{1}{2^6 \alpha k^2} \left[\frac{12 - 32\mu^2 \pm i(38\mu - 8\mu^3)\text{cn}z \text{dn}z}{\text{sn}^4 z} + \frac{(1 + k^2)(3 - 4\mu^2)(\pm i\mu \text{cn}z \text{dn}z - 2)}{\text{sn}^2 z} \right] \\ & + \dots \Big). \end{aligned} \quad (40)$$

Only in the case when all quantities, including in particular the elliptic modulus k , take real values we have $\psi_{\pm}^*(z) = \psi_{\mp}(z)$. In particular, up to the first two leading order the wave functions can be written as

$$\psi_{\pm}(z) \sim \left(\frac{\text{dn}z - k\text{cn}z}{\text{dn}z + k\text{cn}z} \right)^{\pm \frac{\sqrt{\alpha}}{2}} \frac{(\text{dn}z \mp \text{cn}z)^{-\frac{i\mu}{2} - \frac{1}{4}}}{(\text{dn}z \pm \text{cn}z)^{-\frac{i\mu}{2} + \frac{1}{4}}}. \quad (41)$$

This asymptotic solution can be compared to the earlier results about the asymptotic Lamé function obtained by S. Malurkar in the 1930s, and results by H. Müller in the 1960s [16, 14].

In the limit $\alpha \rightarrow \infty, k \rightarrow 0, \mu \rightarrow \nu$ with $\alpha^{\frac{1}{2}} k \rightarrow 2ih^{\frac{1}{2}}$ finite, we recover the unnormalized wave functions which differ some constant terms in the exponent from the asymptotic Mathieu wave functions (28).

The second small energy expansion

Around $z_* = K$ we have $\Lambda = -\alpha k^2 + \text{small correction}$, we set $\Lambda = -\alpha k^2 + \tilde{\Lambda}$ where $\tilde{\Lambda}$ is a small quantity compared with the potential strength αk^2 , and serves as the expansion

parameter. The relation $v_z + v^2 = u - \alpha k^2 + \tilde{\Lambda}$ has an asymptotic solution in the form

$$v(z) = i \sum_{\ell=-1}^{\infty} \frac{v_{\ell}(z)}{(\sqrt{\alpha})^{\ell}}, \quad (42)$$

with

$$\begin{aligned} v_{-1} &= k \operatorname{cn} z, \\ v_0 &= \frac{i}{2} \partial_z \ln \operatorname{cn} z, \\ v_1 &= \frac{1}{8} k \operatorname{cn} z - \frac{4\tilde{\Lambda} + 1 - 2k^2}{8k \operatorname{cn} z} + \frac{3k'^2}{8k \operatorname{sn}^3 z}, \\ v_2 &= -\frac{i}{16k^2} \partial_z \left(\frac{4\tilde{\Lambda} + 1 - 2k^2}{\operatorname{cn}^2 z} - \frac{3k'^2}{\operatorname{cn}^4 z} \right), \quad \text{etc,} \end{aligned} \quad (43)$$

where $k' = \sqrt{1 - k^2}$ is the complementary modulus. After performing integration about the Jacobian elliptic functions we get the wave functions,

$$\begin{aligned} \psi_{\pm}(z) &= \exp \left(\pm \alpha^{\frac{1}{2}} \ln(\operatorname{dn} z + i k \operatorname{sn} z) - \frac{1}{2} \ln \operatorname{cn} z \right. \\ &\quad \pm \frac{i}{2^4 \alpha^{\frac{1}{2}} k} \left[\frac{3 \operatorname{sn} z \operatorname{dn} z}{\operatorname{cn}^2 z} - 2ik \ln(\operatorname{dn} z + i k \operatorname{sn} z) - \frac{8\tilde{\Lambda} - 1 + 2k^2}{k'} \ln \frac{\operatorname{dn} z + k' \operatorname{sn} z}{\operatorname{cn} z} \right] \\ &\quad \left. - \frac{3k'^2 - (4\tilde{\Lambda} + 1 - 2k^2) \operatorname{cn}^2 z}{2^4 \alpha k^2 \operatorname{cn}^4 z} + \dots \right). \end{aligned} \quad (44)$$

The corresponding dispersion relation has been missed for a long time in the literature, motivated by some ideas from quantum gauge theory [10] recently we have derived it by the WKB analysis and a duality argument [11], then we rederive it using the method adopted in this paper [1],

$$\begin{aligned} \tilde{\Lambda} &= i 2 \alpha^{\frac{1}{2}} k \mu + \frac{1}{2^3} (1 - 2k^2) \left(\frac{4\mu^2}{k'^2} + 1 \right) \\ &\quad + \frac{i}{2^5 \alpha^{\frac{1}{2}} k} \left[\frac{(1 - 2k^2)^2}{k'} \left(\frac{4\mu^3}{k'^3} + \frac{3\mu}{k'} \right) + 4k^2 k' \left(\frac{4\mu^3}{k'^3} + \frac{5\mu}{k'} \right) \right] \\ &\quad - \frac{1 - 2k^2}{2^{10} \alpha k^2 k'^2} \left(\frac{80\mu^4}{k'^4} + \frac{136\mu^2}{k'^2} + 9 \right) + \dots \end{aligned} \quad (45)$$

It gives us new asymptotic wave functions,

$$\begin{aligned} \psi_{\pm}(z) &= \exp \left(\pm \alpha^{\frac{1}{2}} \ln(\operatorname{dn} z + i k \operatorname{sn} z) - \frac{1}{2} (\ln \operatorname{cn} z \mp \frac{2\mu}{k'} \ln \frac{\operatorname{dn} z + k' \operatorname{sn} z}{\operatorname{cn} z}) \right. \\ &\quad \pm \frac{1}{2^4 \alpha^{\frac{1}{2}} k} \left[\frac{\pm 8ik'^2 \mu + i(3k'^2 + 4\mu^2) \operatorname{sn} z \operatorname{dn} z}{k'^2 \operatorname{cn}^2 z} + 2k \ln(\operatorname{dn} z + i k \operatorname{sn} z) \right] \\ &\quad - \frac{1}{2^6 \alpha k^2} \left[\frac{k'^2 (12k'^2 + 32\mu^2) \pm (38k'^2 \mu + 8\mu^3) \operatorname{sn} z \operatorname{dn} z}{k'^2 \operatorname{cn}^4 z} \right. \\ &\quad \left. \left. + \frac{(1 - 2k^2)(3k'^2 + 4\mu^2)(\pm \mu \operatorname{sn} z \operatorname{dn} z - 2k'^2)}{k'^4 \operatorname{cn}^2 z} \right] + \dots \right). \end{aligned} \quad (46)$$

They satisfy the property $\psi_{\pm}(-z) = \psi_{\mp}(z)$. In companion with the expression (41) we could write the first two leading order results of the wave functions as

$$\psi_{\pm}(z) \sim \left(\frac{\text{dn}z + ik\text{sn}z}{\text{dn}z - ik\text{sn}z} \right)^{\pm \frac{\sqrt{\alpha}}{2}} \frac{(\text{dn}z \mp k'\text{sn}z)^{-\frac{\mu}{2k'} - \frac{1}{4}}}{(\text{dn}z \pm k'\text{sn}z)^{-\frac{\mu}{2k} + \frac{1}{4}}}. \quad (47)$$

Taking the limit to the Mathieu wave functions (32), we would again encounter the difference of some constant terms which can be absorbed into the normalization constant.

Up to now, everything about the small energy expansions for the Lamé equation is consistent with the known results, although the monodromy relations used to derive the corresponding dispersion relation remain a physics induced conjecture.

4 A connection to N=2 gauge theory

Now we come back to the original motivation which inspired our study the spectral problem of periodic Schrödinger operators, especially for the elliptic potentials. As we have shown in [11, 1], the asymptotic *eigenvalues* of the Mathieu and the Lamé equations are related to the solution of some deformed N=2 supersymmetric Yang-Mills gauge theories in the Nekrasov-Shatashvili limit (NS) [10]. The three asymptotic solutions are precisely in accordance with three different dual descriptions of the low energy effective physics, i.e. the Seiberg-Witten duality [17, 18], in particular the large energy solution is related to the Nekrasov instanton partition function [19].

The large energy asymptotic *wave functions* are related to the instanton partition function of gauge theory with surface operator inserted. The partition function with surface operator extends Nekrasov's localization formula, it is introduced and developed in [20, 21]. The computation can be carried out by the characters developed in [22]. The paper by Alday and Tachikawa [23] gives a detailed study about the relations between the SU(2) gauge theory with surface operator, the SL(2) conformal block and the two-body quantum Calogero-Moser model. In the following, we briefly explain the relation between the gauge theory partition function with surface operator and the asymptotic wave functions (12) and (18).

Let us start from the SU(2) N=2* gauge theory with surface operator, whose partition function takes the following form,

$$Z(a, m, \epsilon_{1,2}, x_{1,2}) = \sum_{k_1, k_2 \geq 0}^{\infty} Z_{k_1, k_2}(a, m, \epsilon_{1,2}) x_1^{k_1} x_2^{k_2}, \quad (48)$$

where a is the scalar v.e.v, m is the mass of adjoint matter, ϵ_1, ϵ_2 are the Ω -deformation and x_1, x_2 are the counting parameters. Written in the exponential form, its pole structure in

the limit $\epsilon_1 \rightarrow 0, \epsilon_2 \rightarrow 0$ is

$$Z = \exp\left(\frac{F}{\epsilon_1 \epsilon_2} + \frac{G}{\epsilon_1}\right). \quad (49)$$

The functions F and G are x_1, x_2 -asymptotic series, their coefficients are functions of $a, m, \epsilon_1, \epsilon_2$ and in particular are regular with respect to ϵ_1, ϵ_2 . There remains an ambiguity of distributing terms, because a term like $\frac{f(a, m, x_1, 2)}{\epsilon_1}$ can be put in either $\frac{F}{\epsilon_1 \epsilon_2}$ or $\frac{G}{\epsilon_1}$. To fix the ambiguity we require that the function F contains only terms with expansion parameters of the form $(x_1 x_2)^n$, while G does not contain such terms. The presence of the surface operator breaks the symmetry between ϵ_1 and ϵ_2 in F and G , therefore F is not exactly the deformed prepotential obtained from the Nekrasov partition function, nevertheless, in the NS limit $\epsilon_2 \rightarrow 0$ they give the same eigenvalue as in (53).

In order to relate gauge theory and the quantum mechanics spectral problem, some manipulations on the function Z are needed. The spectral solution of the Lamé operator is related to the large- a -expansion of instanton partition function (49), in accordance with the large- ν -expansions of the eigenvalue (16) and the eigenfunctions (18). Both $\frac{F}{\epsilon_1 \epsilon_2}$ and $\frac{G}{\epsilon_1}$ contain a -independent terms when expanded as large- a -series, which deserve special attention. These terms are polynomials of x_1, x_2 , and can be represented by the Dedekind eta function and the elliptic theta function,

$$\frac{F}{\epsilon_1 \epsilon_2} = \frac{2(m - \epsilon_1)(m - \epsilon_2)}{\epsilon_1 \epsilon_2} \ln \left(\frac{(x_1 x_2)^{\frac{1}{24}}}{\eta(x_1 x_2)} \right) + \mathcal{O}(a^{-2}), \quad (50)$$

$$\frac{G}{\epsilon_1} = \frac{m - \epsilon_1}{\epsilon_1} \ln \left[\vartheta_4 \left(\frac{i}{4} \ln \frac{x_1}{x_2}, x_1 x_2 \right) \frac{(x_1 x_2)^{\frac{1}{24}}}{\eta(x_1 x_2)} \right] + \mathcal{O}(a^{-1}). \quad (51)$$

To see the connections of functions F, G and the eigenvalue, eigenfunction, we first need to identify the parameters by¹

$$\frac{\pi a}{\epsilon_1} = \omega_1 \nu, \quad \frac{m}{\epsilon_1} = n, \quad x_1 = q^{\frac{1}{2}} e^{-\frac{i\pi}{\omega_1} x}, \quad x_2 = q^{\frac{1}{2}} e^{\frac{i\pi}{\omega_1} x}. \quad (52)$$

The elliptic nome q is the instanton parameter of gauge theory, therefore, the function F is a x -independent q -series which gives the eigenvalue, the function G is a q -series depending on the coordinate x which gives the wave function. The eigenvalue λ in (16) is related to

¹In the paper [12] we actually used the scaled version of elliptic function $\wp(x, \tau) = \wp(x, 1, \frac{\omega_2}{\omega_1})$, this leads to the identification $\frac{2\pi a}{\epsilon_1} = \nu$, this also causes the overall factor of some elliptic functions different from the convention of this paper.

the function F in the limit $\epsilon_2 \rightarrow 0$ by

$$\begin{aligned}
\lambda &= -\nu^2 + \frac{\pi^2}{\omega_1^2} \left[q \frac{\partial}{\partial q} \frac{F|_{\epsilon_2 \rightarrow 0}}{\epsilon_1^2} - \frac{\alpha}{12} (1 - E_2) \right] \\
&= -\nu^2 + \frac{\pi^2}{\omega_1^2} \left[q \frac{\partial}{\partial q} \left(2\alpha \ln \left(\frac{q^{\frac{1}{24}}}{\eta(q)} \right) - \frac{\pi^2 \alpha^2 (q + 3q^2 + 4q^3 + \dots)}{2\nu^2} + \dots \right) - \frac{\alpha}{12} (1 - E_2) \right] \\
&= -\nu^2 + \frac{\alpha^2 (12\zeta_1^2 - g_2)}{48\nu^2} + \mathcal{O}(\nu^{-4}).
\end{aligned} \tag{53}$$

In gauge theory the term $-\nu^2$ is perturbative, hence not included in the instanton partition function. This relation is examined in detail in [11] (see formula (34) in that paper) and in [12], there is a difference of $\alpha\zeta_1 = \alpha \frac{\pi^2}{12\omega_1^2} E_2$ on the right hand side because here we use the shifted potential $\tilde{\varphi}(x)$.

On the other hand, the wave functions (18) is related to the function G in the limit $\epsilon_2 \rightarrow 0$ by

$$e^{-i\nu(x+\omega_2)} \psi_{\pm}(\pm x \pm \omega_2, \nu, \alpha, q) \left[\vartheta_4\left(\frac{\pi x}{2\omega_1}, q\right) \frac{q^{\frac{1}{24}}}{\eta(q)} \right]^{n-1} = \exp\left[\frac{G(a, m, \epsilon_{1,2}, x_{1,2})}{\epsilon_1}\right]_{\epsilon_2 \rightarrow 0}. \tag{54}$$

In the expression we emphasize the parameters used on both sides, and use the property of the large energy wave functions $\psi_-(-x - \omega_2) = \psi_+(x + \omega_2)$. For example, up to the order a^{-2} we have

$$\begin{aligned}
\frac{G(\epsilon_2 = 0)}{\epsilon_1} &= -\frac{m - \epsilon_1}{\epsilon_1} (x_1 + x_2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{3}x_1^3 + x_1^2x_2 + x_1x_2^2 + \frac{1}{3}x_2^3 + \dots) \\
&\quad - \frac{m(m - \epsilon_1)}{2a\epsilon_1} (x_1 - x_2 + x_1^2 - x_2^2 + x_1^3 + x_1^2x_2 - x_1x_2^2 - x_2^3 + \dots) \\
&\quad - \frac{m(m - \epsilon_1)}{4a^2} (x_1 + x_2 + 2x_1^2 + 2x_2^2 + 3x_1^3 + x_1^2x_2 + x_1x_2^2 + 3x_2^3 + \dots) \\
&\quad - \mathcal{O}(a^{-3}).
\end{aligned} \tag{55}$$

Using the relation of parameters given in (52), the first three pieces are summed into three elliptic functions,

$$(n-1) \ln \left[\vartheta_4\left(\frac{\pi x}{2\omega_1}, q\right) \frac{q^{\frac{1}{24}}}{\eta(q)} \right], \quad \frac{i\alpha \partial_x \ln \vartheta_4\left(\frac{\pi x}{2\omega_1}, q\right)}{2\nu}, \quad -\frac{\alpha \partial_x^2 \ln \vartheta_4\left(\frac{\pi x}{2\omega_1}, q\right)}{4\nu^2}, \tag{56}$$

which precisely match the left hand side of (54). The coefficients of the term ν^{-l} , with $l \geq 1$, are linear polynomials of $\partial_x^k \ln \vartheta_4\left(\frac{\pi x}{2\omega_1}, q\right)$ with $k \geq 1$. As the argument of wave functions in (54) is $x + \omega_2$, we have $\tilde{\zeta}(x + \omega_2) = \partial_x \ln \vartheta_4\left(\frac{\pi x}{2\omega_1}, q\right)$, therefore, the coefficients of the term ν^{-l} can be rewritten as linear polynomials of $\partial_x^k \tilde{\zeta}(x + \omega_2)$ with $k \geq 0$, in accordance with the discussion in 2.3. In another form, these coefficients can be rewritten as linear polynomials

of the Jacobi zeta function and its derivatives, because $\partial_x \ln \vartheta_4(\frac{\pi x}{2\omega_1}, q) = \frac{K}{\omega_1} \text{zn}(\frac{Kx}{\omega_1} | k^2)$. See the Appendix of [1] for convention of elliptic functions.

The eigenvalue (16) and the eigenfunction (18) provide an elliptic modular representation for the gauge theory partition function when $\epsilon_2 = 0$. In fact, we observe evidence that even for the case when both deformation parameters are turned on, $\epsilon_1 \neq 0, \epsilon_2 \neq 0$, the instanton partition function with surface operator can be expressed in terms of theta functions. This property indicates the instanton partition function secretly records relations to the elliptic curve. Indeed, this connection can be seen from the point of view of either integrable system or conformal field theory.

In the decoupling limit, the $N=2^*$ gauge theory becomes the pure gauge theory. The corresponding partition function with surface operator can be found in [24], it is related to the asymptotic Mathieu wave functions (12).

Appendix A A matrix that counts divisors of integers

When we take the $a \rightarrow \infty$ limit of the instanton partition with surface operator, only a -independent terms in the functions F and G remain. These terms are represented by two elliptic modular functions,

$$\frac{\eta(q)}{q^{\frac{1}{24}}} = \prod_{n=1}^{\infty} (1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}, \quad (57)$$

$$\begin{aligned} \vartheta_4(\chi, q) &= \prod_{n=1}^{\infty} (1 - 2q^{n-\frac{1}{2}} \cos 2\chi + q^{2n-1})(1 - q^n) \\ &= 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{\frac{n^2}{2}} \cos 2n\chi, \end{aligned} \quad (58)$$

with $\chi = \frac{\pi x}{2\omega_1}$. Had we expanded them as q -series as usual, there might not be interesting things deserve to say. Nevertheless, if we rewrite them in terms of x_1, x_2 as given in (52),

and then expand minus of the logarithm of them as series of x_1, x_2 , we get

$$-\ln \left(\frac{\eta(x_1 x_2)}{(x_1 x_2)^{\frac{1}{24}}} \right) = x_1 x_2 + \frac{3}{2} x_1^2 x_2^2 + \frac{4}{3} x_1^3 x_2^3 + \frac{7}{4} x_1^4 x_2^4 + \frac{6}{5} x_1^5 x_2^5 + 2 x_1^6 x_2^6 + \dots, \quad (59)$$

$$\begin{aligned} -\ln \vartheta_4 \left(\frac{i}{4} \ln \frac{x_1}{x_2}, x_1 x_2 \right) &= x_1 x_2 + \frac{3}{2} x_1^2 x_2^2 + \frac{4}{3} x_1^3 x_2^3 + \frac{7}{4} x_1^4 x_2^4 + \frac{6}{5} x_1^5 x_2^5 + 2 x_1^6 x_2^6 + \dots \\ &\quad + x_1 + x_2 + \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 + \frac{1}{3} x_1^3 + x_1^2 x_2 + x_1 x_2^2 + \frac{1}{3} x_2^3 + \frac{1}{4} x_1^4 + \frac{1}{4} x_2^4 \\ &\quad + \frac{1}{5} x_1^5 + x_1^3 x_2^2 + x_1^2 x_2^3 + \frac{1}{5} x_2^5 + \frac{1}{6} x_1^6 + \frac{1}{2} x_1^4 x_2^2 + \frac{1}{2} x_1^2 x_2^4 + \frac{1}{6} x_2^6 + \dots \\ &= \sum_{i=0, j=0}^{\infty} \Theta_4[i, j] x_1^i x_2^j. \end{aligned} \quad (60)$$

The coefficient matrix $\Theta_4[i, j]$ is a symmetric infinite matrix with all elements positive, as a digest here we present the first 22 dimensions. Notice that the numbers for rows and columns of the matrix begin from 0.

$$\begin{pmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} & \frac{1}{19} & \frac{1}{20} & \frac{1}{21} \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{3}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 1 & \frac{4}{3} & 1 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 1 & \frac{7}{4} & 1 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & 0 & 0 & 0 & 1 & \frac{6}{5} & 1 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & 0 & 0 & \frac{1}{3} & \frac{1}{2} & 1 & 2 & 1 & \frac{1}{2} & \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 1 & \frac{8}{7} & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{8} & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 1 & \frac{15}{8} & 1 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{9} & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 1 & \frac{13}{9} & 1 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 & 0 & 0 \\ \frac{1}{10} & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{2} & 1 & \frac{9}{5} & 1 & \frac{2}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & \frac{1}{10} & 0 \\ \frac{1}{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{12}{11} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & \frac{7}{3} & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{14}{13} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{14} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{12}{7} & 1 & \frac{1}{2} & 1 & 0 & 0 & 0 & \frac{1}{7} & 0 \\ \frac{1}{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 1 & \frac{8}{5} & 1 & 1 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & 0 \\ \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 1 & \frac{31}{16} & 1 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ \frac{1}{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{18}{17} & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{18} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{3} & \frac{1}{2} & 1 & \frac{13}{6} & 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{19} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{20}{19} & 1 & 1 & 0 \\ \frac{1}{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10} & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{4} & 0 & \frac{1}{2} & 1 & \frac{21}{10} & 1 \\ \frac{1}{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & 0 & 0 & 0 & \frac{1}{3} & 0 & 1 & \frac{32}{21} \end{pmatrix}$$

A gaze reveals that the matrix $\Theta_4[i, j]$ contains information about an elementary fact of number theory. Given a positive integer $n \geq 1$, there are $d(n)$ positive divisors denoted as the set $\{d_1, d_2, d_3, \dots, d_{d(n)}\}$, in a decreasing order $d_1 > d_2 > \dots > d_{d(n)}$. Among the divisors, we have $d_1 = n, d_{d(n)} = 1$. The divisor function is defined by $\sigma_k(n) = \sum_i d_i^k$. The nonzero

diagonal elements of the matrix, which are the coefficients of the q -series for $\ln \frac{q^{\frac{1}{24}}}{\eta(q)}$, are $\sigma_{-1}(n)$ for all positive integers, $\Theta_4[n, n] = \sigma_{-1}(n)$. Then the expansion of the logarithm of eta function is $\ln \eta(q) = \frac{1}{24} \ln q - \sum_{n=1}^{\infty} \sigma_{-1}(n) q^n$. A further derivative gives the expression for the second Eisenstein series, $E_2(q) = 24q \partial_q \ln \eta(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$, where the relation $n\sigma_{-1}(n) = \sigma_1(n)$ is used. The nonzero off-diagonal elements of the the matrix are precisely the inverse of the divisors of integers. If we focus on the lower triangular part, in the n -th row the nonzero numbers are $\{d_1^{-1}, d_2^{-1}, d_3^{-1}, \dots, d_{d(n)}^{-1}\}$ and their sum:

$$\sum_{i=1}^{d(n)} \frac{1}{d_i} = \Theta_4[n, n]. \quad (61)$$

For example, in the 18-th row we have the divisor function $\sigma_{-1}(18)$,

$$\frac{1}{18} + \frac{1}{9} + \frac{1}{6} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} = \frac{13}{6}. \quad (62)$$

This fact indicates a relation of the eta function and the theta function expanded as in (59),(60): while the logarithm of eta function knows the infinite sequence of numbers $1, \frac{3}{2}, \frac{4}{3}, \frac{7}{4}, \dots$, the logarithm of theta function diagnoses where they come from.²

If a divisor d_i is a composite number, we can always move up to the d_i -th row to find its divisors other than 1 and d_i . Eventually, we could arrive at the unique prime factorization for the integer $n = \prod_{i=1}^s p_i^{\nu_i}$. A well known fact of number theory asserts that the number of nonzero elements from $\Theta_4[n, 0]$ to $\Theta_4[n, n-1]$ in the n -th row is given by $d(n) = \prod_{i=1}^s (\nu_i + 1)$. The n -th column tells the same story.

In fact, the story goes a bit further. The large ν expansion of wave function (instanton partition function) involves higher order χ -derivatives of $\ln \vartheta_4(\chi, q)$. All elliptic functions in (54), (55) are made out of $\partial_{\chi}^k \ln \vartheta_4(\chi, q)$, with $k \geq 0$, they count the $(k-1)$ -th power of divisors. More precisely, the coefficient matrix of $-(\frac{i}{2})^k \partial_{\chi}^k \ln \vartheta_4(\chi, q)$ expanded as x_1, x_2 -series counts d_i^{k-1} for all positive integers. The discussion above is just about the first case $k = 0$.

This might be a folklore of number theory, nevertheless, it is strange instanton knows it.

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²It is the same in spirit for the situation in gauge theory: while the instanton partition function *without* surface operator knows the eigenvalue expansion, it is the instanton partition function *with* surface operator tells the whole story.

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